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THEORY OF THE STEADY LAMINAR BUOYANT FLOW ABOVE A LINE HEAT SOURCE IN A FLUID OF LARGE PRANDTL NUMBER AND TEMPERATURE-DEPENDENT VISCOSITY

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Abstract—Schuh's [1] theory of steady laminar flow above a line heat source in a fluid with $Pr = 0{\cdot}7$ is extended to fluids such as heavy oils. The solution is valid for all fluids of high Prandtl number, regardless of whether the viscosity is temperature-dependent or not.

Résumé—Le théorie de Schuh de l'écoulement laminaire permanent, au-dessus d'une source de chaleur linéaire, dans un fluide dont $Pr = 0{,}7$, a été étendue aux fluides tels que les huiles lourdes. La solution est valable pour tous les fluides à grand nombre de Prandtl, que leur viscosité varie ou non avec la température.

Zusammenfassung—Die Theorie von Schuh der stationären Laminarströmung über einer linienförmigen Wärmequelle in einer Flüssigkeit mit $Pr = 0{,}7$ wird auf andere Flüssigkeiten wie Schweröle erweitert. Die Lösung gilt für alle Flüssigkeiten hoher Prandtlzahl, gleichgültig ob ihre Viskosität temperaturabhängig ist oder nicht.

Аннотация—Приложение теории Шаха 1 об установившемся ламинарном потоке над линейным источником тепла в жидкости с $Pr = 0{,}7$ распространяется на такие жидкости, как тяжелые масла. Зто решение справедливо для всех жидкостей с большими числами Прандтля независимо от того, зависит ли вязкость от температуры.

NOTATION

- c — specific heat of the fluid at constant pressure;
- f — $\psi/\Lambda^{1/5}\nu_\infty x^{3/5}$;
- g — gravitational acceleration;
- k — thermal conductivity of fluid;
- $\dot{m}'(x)$ — vertical mass flux per unit length;
- Pr_∞ — $c_\infty\mu_\infty/k_\infty$, Prandtl number in the undisturbed fluid;
- $\dot{q}'$ — rate of heat transfer from unit length of source;
- t — fluid temperature;
- u — fluid velocity in vertical direction;
- v — fluid velocity in horizontal direction;
- x — vertical height above the heat source;
- y — horizontal distance from the plane of symmetry;
- β — volumetric thermal expansion coefficient;
- ϵ — a small number;
- η — $\Lambda^{1/5}/x^{2/5}\int_0^y \rho/\rho_\infty \cdot dy$;
- θ — $(t - t_\infty)g\beta x^{3/5}/\nu_\infty^2\Lambda^{4/5}$;
- Λ — $\dot{q}'g\beta/c_\infty\rho_\infty\nu_\infty^3$;
- μ — dynamic viscosity of the fluid;
- ν — kinematic viscosity of the fluid;
- ξ — $-f''/f_0'^{3/2}$;
- ρ — fluid density;
- τ — $(t - t_\infty)$;
- ψ — stream function;
- χ — f'/f_0'.

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Subscripts

0 in the central plane;

∞ in the undisturbed fluid;

(0), (1), (2), . . . zeroth, first, second, . . . approximations.

1. INTRODUCTION

1.1 *The problem*

HEAVY oil stored in a tank is sometimes heated by horizontal steam pipes placed near the tank floor. When calculating the temperature distribution which will be found in the tank after heating has continued for a given time, it is important to know how much oil is convected upwards in the plume of heated fluid which forms itself above the pipe.

The present paper provides a theory for the steady laminar flow in a plume above a long horizontal heat source in a large tank of stagnant oil. Although this system is idealized, as compared with tanks of finite size containing several heating pipes and with slowly changing bulk temperature, it is thought that the analysis may form a first step towards predicting what happens in practice.

1.2 *Previous work on the subject*

Schuh [1] has developed a theory for the laminar buoyant flow above a two-dimensional heat source in a fluid of Prandtl number equal to 0·7. Since oils have Prandtl numbers which are very much greater than unity, we shall rework Schuh's problem for the case of infinite Prandtl number.

In Schuh's work the transport properties of the fluid were taken to be uniform. Since oils have viscosities which depend steeply upon temperature, we shall take account of this dependence. It will be shown that this involves trivial difficulty when the Prandtl number is large.

1.3 *Outline of main result*

The main result of the analysis is given in equation (19) below; this shows that the upward mass flow rate per unit length is proportional to the one-fifth power of the heat flux per unit length, the three-fifths power of the distance above the heat source, and the two-fifths power of the bulk viscosity of the oil.

2. MATHEMATICAL ANALYSIS

2.1 *Equations and boundary conditions*

The velocity and temperature distributions in the fluid are described by the following equations:

Mass continuity

$$(\rho u)_x + (\rho v)_y = 0 : \frac{u\rho}{\rho_\infty} = \psi_y : \frac{v\rho}{\rho_\infty} = \psi_x \qquad (1)$$

Momentum

$$\rho u u_x + \rho v u_y = \rho\beta g\tau + (\mu u_y)_y \qquad (2)$$

Energy

$$\rho u(c\tau)_x + \rho v(c\tau)_y = (k\tau_y)_y. \qquad (3)$$

Dimensional analysis indicates that the variables are related by:

$$\frac{\psi}{\nu_\infty \Lambda^{1/5} x^{3/5}} \equiv f(\eta, Pr_\infty)$$

$$\frac{\tau g\beta x^{3/5}}{\nu_\infty \Lambda^{4/5}} \equiv \theta(\eta, Pr_\infty) \qquad (4)$$

$$\eta \equiv \frac{\Lambda^{1/5}}{x^{2/5}} \cdot \int_0^y \frac{\rho}{\rho_\infty} \cdot \mathrm{d}y.$$

Equations (1), (2) and (3) are transformed by equations (4) into the following ordinary differential equations, which are identical to Schuh's [1] equations for the case of constant properties:

$$\left[\frac{\rho\mu}{\rho_\infty \mu_\infty} \cdot f''\right]' + \tfrac{3}{5} ff'' + \theta - \tfrac{1}{5} f'^2 = 0 \qquad (5)$$

$$\left[\frac{\rho k}{\rho_\infty k_\infty} \cdot \theta'\right]' + \tfrac{3}{5} Pr_\infty \left[\frac{c}{c_\infty} f\theta\right]' = 0, \qquad (6)$$

where the prime denotes differentiation with respect to η, and other notation is given in section 5 below.

The boundary conditions may be expressed as:

$$\left.\begin{aligned} \eta = 0 &: f'' = 0 : \theta' = 0 \\ \eta = \infty &: f' = 0 : \theta = 0 \end{aligned}\right\} \qquad (7)$$

and, in addition,

$$\int_0^\infty f'\theta \mathrm{d}\eta = \tfrac{1}{2}. \qquad (8)$$

The differential equations involve the usual boundary-layer assumptions. In addition it is supposed that there is no influence of the force exerted on the fluid at the heat source. These assumptions are probably fulfilled in practice at a height which is sufficiently far above the steam pipe; they are responsible for the reduction of the partial differential equations to ordinary ones, i.e. for the existence of "similar" velocity and temperature profiles. Equation (8) equates the enthalpy flux through any horizontal plane to the rate of energy release from the heat source.

2.2 *Equations and boundary conditions for large Prandtl number*

The following argument shows that a solution to equations (5) and (6) may be obtained with little difficulty for the case of large Prandtl number.

Step (*i*). When the Prandtl number is high, the thermal boundary layer is much thinner than the velocity layer. (This fact, which is familiar from other boundary-layer studies, is proved *a posteriori* in section 2.3.)

Step (*ii*). Since the vertical velocity in the thin thermal layer is approximately uniform in a horizontal plane, we can write:

$$f' \approx f'_0 : \therefore f \approx \eta f'_0. \qquad (9)$$

We assume that c, ρ, k are substantially invariant with temperature for oils, since it is the rapid variation in viscosity with temperature which is of major concern ($\mu \sim t^5$ approx.). Equation (6) now assumes the asymptotic form:

$$\theta'' + \tfrac{3}{5} Pr_\infty f'_0 [\eta\theta]' = 0. \qquad (10)$$

Step (*iii*). A further consequence of Step (i) is that θ can be taken to be equal to zero in the region surrounding the thermal boundary layer, $\eta > \epsilon$, where ϵ is a small quantity. Equation (5) can therefore be re-written as:

$$0 \leqslant \eta \leqslant \epsilon : \left[\frac{\mu}{\mu_\infty} f''\right]' + \tfrac{3}{5} f'_0 \eta f'' + \theta - \tfrac{1}{5} f'^2_0 = 0 \qquad (11)$$

$$\epsilon < \eta < \infty : f''' + \tfrac{3}{5} f f'' - \tfrac{1}{5} f'^2 = 0. \qquad (12)$$

Step (*iv*). The boundary conditions for (11) and (12) become:

$$\left.\begin{array}{l} \eta = 0 : \; f'' = 0 \\ \eta = \epsilon : \; f'' = f''_\epsilon \end{array}\right\} \qquad (13)$$

$$\left.\begin{array}{l} \eta = \epsilon : f = 0 : f' = f'_0 : f'' = f''_\epsilon \\ \eta = \infty : f' = 0 \end{array}\right\} \qquad (14)$$

Conclusion. Our problem is thus reduced to the solutions of: equation (10) subject to the conditions (7) and (8), in the domain $0 \leqslant \eta \leqslant \epsilon$; equation (11) subject to the conditions (13) in the same domain; and equation (12) subject to the conditions (14) in the domain $\eta > \epsilon$. Only the last equation has to be solved numerically as will now be seen.

2.3 *Solution of the differential equations*

Equation (10) and its appropriate conditions are readily seen to be solved by:

$$\theta = \left(\frac{3Pr_\infty}{10\pi f'_0}\right)^{\frac{1}{2}} . \exp\left(-\frac{3Pr_\infty f'_0 \eta^2}{10}\right). \qquad (15)$$

Were it not that f'_0 is an as yet undetermined number, equation (15) would completely describe the temperature distribution in the plume. It should be noted that the equation confirms that θ falls off rapidly with η when Pr_∞ is large. Correspondingly, the maximum value of θ, which occurs when $\eta = 0$, increases as the square root of Prandtl number.

Equation (11). By integrating with respect to η, we obtain:

$$\left[\frac{\mu}{\mu_\infty} f''\right]_0^\epsilon + \tfrac{3}{5} f'_0 [\eta f' - f]_0^\epsilon + \int_0^\epsilon \theta \, d\eta - \tfrac{1}{5} f'^2_0 \epsilon = 0. \qquad (11a)$$

Introducing the boundary conditions (13) and the integral condition (8), and noting that ϵ is a small quantity, we see that equation (11a) reduces to:

$$f'_0 f''_\epsilon + \tfrac{1}{2} = 0. \qquad (16)$$

This result is independent of the mode of variation of viscosity with temperature.

Equation (12). Equation (16) can be used to displace f''_ϵ from the conditions on equation (12), which can now be regarded as defining an eigenvalue problem, f'_0 being the quantity to be determined.

The procedure for solution of this problem is indicated in the Appendix; the equation has been solved by an iterative quadrature method after the manner of Crocco [2]. For present purposes it suffices to note the result:

$$f_0' = 0{\cdot}9335. \tag{17}$$

Also calculated was f_∞, which took the value:

$$f_\infty = \int_0^\infty f' \, d\eta = 1{\cdot}346. \tag{18}$$

The variation of f' with η corresponding to the solution is shown in Fig. 1 as the curve marked: $Pr_\infty = \infty$.

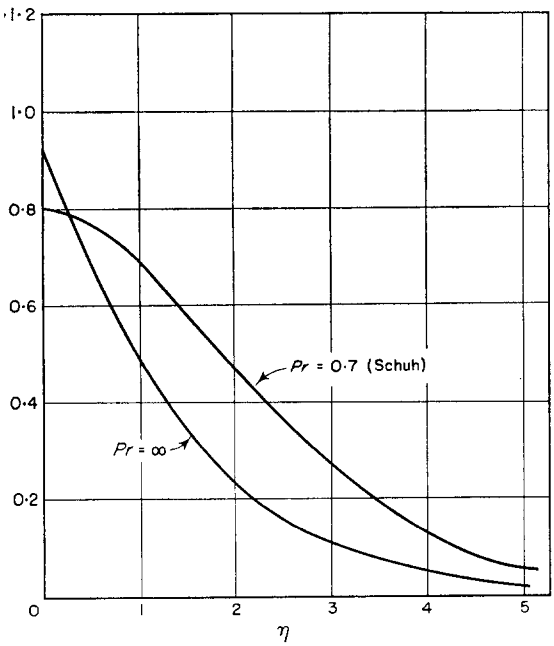

FIG. 1. Non-dimensional velocity distributions above heated line source in laminar steady flow.

3. DISCUSSION

3.1 *Interpretation of the result*

Equation (18), when translated into physical quantities, signifies:

$$\dot{m}'(x) = 2\int_0^\infty \rho u \, dy = 2{\cdot}69\,[\dot{q}' g\beta \rho_\infty^2 \mu_\infty^2 x^3 / c_\infty]^{1/5}. \tag{19}$$

Thus the mass flow rate in the plume increases in accordance with the one-fifth power of the heating rate, the two-fifths power of the viscosity in the bulk of the fluid, and the three-fifths power of the distance above the heat source.

Neither the thermal conductivity nor the temperature-dependence of the viscosity have any influence on this result, the reason being that, when the Prandtl number is large, the region of non-uniform temperature is very thin and is moreover concentrated in a region of small shear stress.

Equation (17) correspondingly signifies that the upward velocity of fluid at the symmetry plane of the plume, u_0, is given by:

$$u_0 = 0{\cdot}934\ [\dot{q}' g\beta / c_\infty]^{2/5} \cdot [x/\rho_\infty \mu_\infty]^{1/5}. \tag{20}$$

This equation implies that the peak velocity in the plume increases with the one-fifth power of the height above the source, and with the two-fifths power of the strength of the heat source.

Equations (15) and (17), taken together, signify that the greatest temperature at any level in the plume is given by:

$$t - t_\infty = 0{\cdot}32\left[\frac{\dot{q}'}{c_\infty}\right]^{4/5} \cdot \frac{[c_\infty \mu_\infty / k_\infty]^{1/2}}{[g\beta \rho_\infty^2 \mu_\infty^2]^{1/5} x^{3/5}}. \tag{21}$$

Hence we conclude that the maximum temperature in the plume falls off as the minus three-fifths power of the height; it is increased slightly (as the one-tenth power) by an increase in viscosity, but reduced by an increase in thermal conductivity.

It might be noted that the present theory is most easily checked experimentally by way of a measurement of the peak temperature in the plume.

3.2 *Comparison with Schuh's* [1] *result for* $Pr = 0{\cdot}7$

Figure 1 also contains, as a curve marked $Pr = 0{\cdot}7$, the result obtained by Schuh [1] for uniform transport properties. It is seen that the finite Prandtl number results in only modest changes in the $f'(\eta)$ curve, which now exhibits a rounded peak because the buoyancy forces now operate over a region of finite thickness.

It may be mentioned that Schuh found that the constant in the equation for $(t - t_\infty)$ was 0·37, instead of the 0·32 indicated above. This results, no doubt, from the fact that for $Pr = 0{\cdot}7$ the average velocity of the heated fluid is less than the maximum velocity in the plume, instead of being nearly equal to it as when the Prandtl number is very large.

REFERENCES

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2. L. Crocco, Una carateristica transformazione delle equazione dello strato nei gas. *Atto di Guidonia*, No. 7 (1939).

APPENDIX

Procedure for solving equation (7) by iterative quadrature

The problem to be solved is the determination of f_0' and f_∞ from the solution to the equation:

$$f''' + \tfrac{3}{5}ff'' - \tfrac{1}{5}f'^2 = 0 \qquad \text{(A.1)}$$

with the boundary conditions:

$$\eta = 0: \ f = 0 : f' = f_0' : f'' = -\frac{1}{2f_0'}$$
$$\eta = \infty : f' = 0. \qquad \text{(A.2)}$$

We introduce a transformation similar to that of Crocco [2]:

$$\xi \equiv -f'' f_0'^{-3/2} \qquad \text{(A.3)}$$

$$\chi \equiv f'/f_0'. \qquad \text{(A.4)}$$

Equation (A.1) then reduces to the first-order form:

$$\xi \frac{d\xi}{d\chi} + \tfrac{3}{5}\xi \int_1^\chi \frac{\chi}{\xi}\, d\chi - \tfrac{1}{5}\chi^2 = 0 \qquad \text{(A.5)}$$

while the boundary conditions reduce to:

$$\chi = 1 : \xi = \tfrac{1}{2} f_0'^{-5/2} \qquad \text{(A.6)}$$

$$\chi = 0 : \xi = 0. \qquad \text{(A.7)}$$

Equation (A.5) can be integrated formally to yield, after substitution of equation (A.7):

$$\tfrac{1}{2}\xi^2 = \tfrac{3}{5}\int_0^\chi \xi \int_\chi^1 \frac{\chi}{\xi}\, d\chi\, d\chi + \tfrac{1}{15}\chi^3. \qquad \text{(A.8)}$$

Now equation (A.8) can be used as an iteration formula for the determination of $\xi(\chi)$. The zero'th approximation $\xi_{(0)}$ is obtained by neglecting the quadrature expression entirely. Thus:

$$\xi_{(0)} = (\tfrac{2}{15}\chi^3)^{1/2}. \qquad \text{(A.9)}$$

The first approximation $\xi_{(1)}$ is obtained by substituting $\xi_{(0)}$ for ξ in the quadrature of (A.8). Thus:

$$\tfrac{1}{2}\xi_{(1)}^2 = \tfrac{3}{5}\int_0^\chi \chi^{3/2} \int \chi^{-1/2}\, d\chi\, d\chi + \frac{\chi^3}{15}$$
$$= \tfrac{12}{25}\chi^{5/2} - \tfrac{1}{3}\chi^3 \qquad \text{(A.10)}$$

which is a better approximation.

Further approximations involve numerical work. We here merely tabulate the resulting values for ξ^2 at $\chi = 1$, so that the rapidity of convergence can be seen. (See Table 1.)

The fifth approximation was regarded as sufficiently exact for our purposes.

Table 1

Approximation	Zero'th	First	Second	Third	Fourth	Fifth
ξ^2 at $\chi = 1$	0·133	0·293	0·334	0·346	0·3519	0·3526
f_0'	1·136	0·969	0·944	0·937	0·9339	0·93356